Nonlinear anomalous ferromagnetic Faraday effect

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Two electromagnetic waves with the same frequencies and different polarizations may propagate together in a saturated ferrite. We investigate how the simultaneous presence of the two waves affects their modulation. It is found that their evolution is governed by two independent nonlinear Schrödinger equations. A phase factor corresponding to a weak interaction is created: it is interpreted as a nonlinear Faraday effect. Then, for high frequencies, we build a perturbative method adapted to the study of the nonlinear Faraday effect. The angle of rotation of the polarization is calculated and expressed in terms of generalized Stokes parameters of the waves. For the linear case, it is well known that the Faraday effect has the following important property: when the wave is reflected back and passes through the sample of ferrite in the opposite direction, the Faraday effect rotates the polarization of the reflected wave in the same way as the incident one; thus the additional rotation angle is added to the first one instead of canceling it. We describe a normal Faraday effect, which has the same property, and an anomalous effect, for which the rotation is canceled in the same conditions. The normal effect is proportional to the energy density of the incident wave, and the anomalous effect to the difference of intensity between the two elliptic polarizations. Finally, we discuss ways of making evident these higher-order effects.

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I. INTRODUCTION

The problem of propagation of electromagnetic waves in ferromagnetic dielectrics is a very complex matter that has given rise to many studies. The linear study of plane waves, which can be found, e.g., in [1], uses a simple model (see below) assuming that the medium is infinite and neglecting inhomogeneous exchange interaction, anisotropy, and damping. This model assumes also that the medium is immersed in a constant exterior magnetic field strong enough to magnetize it to saturation.

Corrections to this model may be done to take into account the neglected factors, in particular the damping. Also, for experiments, the study of propagation in a waveguide instead of an infinite medium is important [2]. Our aim is not to refine the model in such a way, but to study nonlinear effects on the simple model.

At a given frequency, two plane waves may propagate in a magnetized ferromagnetic dielectric. Both have elliptic polarizations (circular if the exterior field is parallel to the propagation direction), completely defined by the medium, the exterior field, and the wave vector. In previous papers [3,4], we have studied the nonlinear evolution of one of these plane waves, assumed to propagate alone, and have shown that it obeys the nonlinear Schrödinger (NLS) equation. This led to the characterization of a Benjamin-Feir-type instability: in the particular case of propagation parallel to the exterior field, the stability of the wave depends only on its polarization.

But in usual experiments, an incident monochromatic wave entering a sample of ferrite would rather be linearly polarized. It breaks up inside the medium into the sum of two waves with different circular (or elliptic) polarization. In a magnetized ferrite, these two polarizations propagate with different phase velocities; thus after some time a phase shift appears between the two polarizations. At the exit of the sample, the two elliptic polarizations recombine and the wave is again linearly polarized. Because of the phase shift, the direction of the linear polarization has rotated for a certain angle. This phenomenon is well known as the Faraday effect [2,5].

An important particularlity of this effect is the following: if the wave is reflected back and passes through the same sample in the opposite direction, the polarization rotates again. But this second rotation adds to the first one instead of canceling it and the polarization of the outgoing reflected wave is not parallel to the polarization of the incoming incident wave.

As written before, the effects of nonlinearity on the propagation of a single wave have been studied. Are the corresponding results still valid if the two polarizations propagate together? In order to answer this question, we first look at the nonlinear interaction between the two waves of the same frequency and different polarization that may propagate in a saturated ferrite at the same time, space, and amplitude scales as to obtain the NLS equation. We will see that the NLS equation is still valid and find a leading term of nonlinear Faraday rotation, proportional to the amplitude of the wave.

Although the preceding calculus makes evident the existence of a nonlinear contribution to the Faraday effect, it is not appropriate for the computation of the corresponding rotation angle. Thus we build a perturbative scheme appropriate for the study of the nonlinear ferromagnetic Faraday effect. Using this method we can compute an explicit expression for the angle of rotation of the polarization. The previous leading term is found again and stated precisely; it has the same behavior as the linear term when the wave is reflected back and is therefore called "normal." An additional term is found that is anomalous in the following sense: when the wave is reflected back, it does not have the same property as the linear Faraday rotation, but cancels the corresponding term in the rotation of the incident wave. This term is proportional to the generalized Stokes parameter s_3 of the wave.

The reader will note that we study here a *nonresonant* interaction between two waves instead of considering a resonant interaction, as it is usually the case. A resonant interaction leads for three waves to the three-wave resonant interaction, completely integrable system [6], for which there exists an analog with two waves. This is not expected here.

On the other hand, a more complete description of the nonresonant interaction of these two waves could be done either by studying the evolution of terms of higher order in the calculus involved here in Sec. II or by using a stretching that corresponds to an incident wave of higher power. Such a study is not the purpose of this paper.

II. THE APPROXIMATION LEADING TO THE NLS EQUATION: THE NORMAL NONLINEAR FARADAY EFFECT

A. Model and perturbation method

We consider a ferromagnetic dielectric and assume that the electric part of the Maxwell equations in this medium is linear, so that the magnetization density $\vec{M}$ and the magnetic field $\vec{H}$ verify

$$-\vec{\nabla}(\vec{\nabla}\cdot\vec{H})+\Delta\vec{H}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\vec{H}+\vec{M}) \tag{1}$$

where $c=1/\sqrt{\hat{\epsilon}\mu_0}$ is the speed of light based on the dielectric constant $\hat{\epsilon}$ of the medium. Neglecting damping, the time evolution of $\vec{M}$ is given by the torque equation

$$\frac{\partial\vec{M}}{\partial t}=-\mu_0\delta\vec{M}\times\vec{H}\ , \tag{2}$$

where δ is the gyromagnetic ratio. In this formula, the magnetic field $\vec{H}$ could be replaced by an effective field to take account of the anisotropy and the inhomogeneous exchange interaction, but we will not do that here (for more details and a justification of this approximation see [3,4]).

Rescaling $\vec{M}$, $\vec{H}$, and t into $(\delta\mu_0/c)\vec{M}$, $(\delta\mu_0/c)\vec{H}$, and ct, Eqs. (1) and (2) become

$$-\vec{\nabla}(\vec{\nabla}\cdot\vec{H})+\Delta\vec{H}=\frac{\partial^2}{\partial t^2}(\vec{H}+\vec{M})\ , \tag{1$'$}$$

$$\frac{\partial\vec{M}}{\partial t}=-\vec{M}\times\vec{H}\ . \tag{2$'$}$$

In order to describe the interaction between two waves propagating in the same direction, with pulsations ω_1 and ω_2 and wave numbers k_1 and k_2, respectively, we introduce the phases

$$\varphi_j=k_jx-\omega_jt\quad(j=1,2) \tag{3}$$

and expand the vector $\vec{M}$ in the form

$$\vec{M}=\sum_{n\in\mathbb{Z}^2}\vec{M}_n e^{in\cdot\varphi} \tag{4}$$

with, for $n=(n_1,n_2)$,

$$n\cdot\varphi=n_1\varphi_1+n_2\varphi_2\ . \tag{5}$$

$\vec{M}$ is expected to be real, thus $\vec{M}_{-n}=\vec{M}_n^*$ for each $n\in\mathbb{Z}^2$ (an asterisk denotes complex conjugate), and the $\vec{M}_n$ vary slowly with respect to the phases φ_j.

Two waves of the same frequency and different polarizations may propagate in the ferromagnet. These are the two waves under consideration; in other words, we take $\omega_1=\omega_2=\omega$ and $k_1\neq k_2$. The ratio k_1/k_2 is generally irrational. We assume that that is the case and thus all the functions $e^{in\cdot\varphi}$ will be linearly independent. This assumption means that there is no couple of harmonics $\vec{M}_n e^{in\cdot\varphi}$ and $\vec{M}_{n'}e^{in'\cdot\varphi}$, $n\neq n'$, that will resonate together: we are considering here a nonresonant interaction.

Let us now expand each $\vec{M}_n$ in powers series of a small parameter ε,

$$\vec{M}_n=\vec{M}_n^0+\varepsilon\vec{M}_n^1+\varepsilon^2\vec{M}_n^2+\cdots\ . \tag{6}$$

We restrict ourselves to one space dimension and assume that each quantity $\vec{M}_n^k$ is a function of the three stretched coordinates ξ, T, and τ defined by

$$\frac{\partial}{\partial x}=\varepsilon\frac{\partial}{\partial\xi},\quad \xi=\varepsilon x$$
$$\frac{\partial}{\partial t}=\varepsilon\frac{\partial}{\partial T}+\varepsilon^2\frac{\partial}{\partial\tau},\quad T=\varepsilon t,\quad \tau=\varepsilon^2 t\ . \tag{7}$$

This two-time-scale expansion is the same as the usual one that leads to the NLS equation (see [7], Sec. 8.1). The latter reads

$$\xi=\varepsilon(x-Vt),\quad \tau=\varepsilon^2 t \tag{8}$$

and if we assume for the moment that we are looking simply for a single wave, the second condition obtained through the perturbation calculus (the first one is the dispersion relation) is that V must be the group velocity V_G of the wave. Using the form (7) instead of (8) to define the stretched coordinates, this condition takes the form

$$\frac{\partial g}{\partial T}+V_G\frac{\partial g}{\partial\xi}=0\ , \tag{9}$$

where g is the unknown function proportional to the amplitude of the wave under consideration. Solving (9), one finds that g is allowed to be a function of the variables ξ and τ defined by (8) only. Thus (7) defines really the same approximation as in [3,4]. To study a two-wave interaction, it is necessary to use the form (7) because two different group velocities will be found. An analogous expansion is made for $\vec{H}$.

$\vec{H}_{0,0}^0$ and $\vec{M}_{0,0}^0$ represent the exterior constant field and the corresponding magnetization (all effects due to the finite size of the sample are neglected). We assume that $\vec{H}_n^0=\vec{M}_n^0=0$ for $n\neq(0,0)$. $\vec{H}_{1,0}^1,\vec{M}_{1,0}^1$ and $\vec{H}_{0,1}^1,\vec{M}_{0,1}^1$ represent the amplitudes of the two waves under consideration. Others terms of order ε^1 are assumed to be zero. We will assume that these two amplitudes have finite, very slowly varying limits $\rho_1(\tau)$ and $\rho_2(\tau)$ as $\xi \to -\infty$. All other harmonics will be assumed to vanish as $\xi \to -\infty$, except the ones that cannot have any limit, for which we will take integration constants equal to zero. Putting expansions (4), (6), and (7) into the basic equations (1′) and (2′), collecting powers of ε, and using the fact that the $e^{in\cdot\varphi}$ are linearly independent, we can solve the perturbative scheme order by order.

B. Results of perturbative calculus

This perturbative calculus is fruitful but quite long and some technical difficulties are not relevant for the present problem. Thus we have chosen to present here only the results that concern the study of the Faraday effect, leaving proofs and explicit expressions of the coefficients for further publication.

At order ε^0, we find that $\vec{H}^0_{0,0}$ and $\vec{M}^0_{0,0}$ must be parallel. Let

$$\vec{M}^0_{0,0} = \vec{m}, \quad \vec{H}^0_{0,0} = \alpha \vec{m} \ . \tag{10}$$

We choose the y axis such that

$$\vec{m} = \begin{bmatrix} m_x \\ m_t \\ 0 \end{bmatrix}$$

and define m and θ through

$$m_x = m\cos\theta, \quad m_t = m\sin\theta \ . \tag{11}$$

m is the magnetization saturation of the ferromagnetic; it is the only dimensional parameter left in the problem. θ is the angle between the propagation direction and the exterior field and α measures the strongness of the latter. Together with the pulsation ω, these are the parameters of our problem.

At order ε^1, we find that

$$\begin{aligned} \vec{H}^1_{1,0} &= g_1(\xi,T,\tau)\vec{h}^{\,1}_{1,0} \ , \\ \vec{M}^1_{1,0} &= g_1(\xi,T,\tau)\vec{m}^{\,1}_{1,0} \ , \\ \vec{H}^1_{0,1} &= g_2(\xi,T,\tau)\vec{h}^{\,1}_{0,1} \ , \\ \vec{M}^1_{0,1} &= g_2(\xi,T,\tau)\vec{m}^{\,1}_{0,1} \ , \end{aligned} \tag{12}$$

where $\vec{h}^{\,1}_{1,0}$, $\vec{m}^{\,1}_{1,0}$, $\vec{h}^{\,1}_{0,1}$, and $\vec{m}^{\,1}_{0,1}$ are constants polarization vectors and g_1 and g_2 unknown functions. We find also that both (ω,k_1) and (ω,k_2) must verify the dispersion relation

$$\left[1+\alpha\left[1-\frac{k^2}{\omega^2}\right]\right]^2 m_x^2 + \left[1-\frac{k^2}{\omega^2}\right]\left[1+\alpha\left[1-\frac{k^2}{\omega^2}\right]\right](1+\alpha)m_t^2 = \left[1-\frac{k^2}{\omega^2}\right]^2\omega^2 \ . \tag{13}$$

This dispersion relation is of degree 2 in k^2 and thus there are two real positive k solutions for every positive ω, except $m\sqrt{\alpha(\alpha+\sin^2\theta)} \le \omega \le m(1+\alpha)$. We assume that ω lies outside of this interval and we choose the two corresponding values of k for the wave numbers k_1 and k_2 of the two waves under consideration.

At order ε^2, for $|n|=1$, one finds as a solvability condition

$$\frac{\partial}{\partial T}g_j + V_j\frac{\partial}{\partial \xi}g_j = 0 \quad (j=1,2) \ , \tag{14}$$

where V_j is the group velocity of the wave (ω_j,k_j). Equation (14) shows that each g_j is a function of the variables $\xi_j = \xi - V_j T$ and τ only.

The quantities representing the second-order terms in the expansion of the fields are eliminated by using the fact that g_j is a function of ξ_j and τ only, for each $j=1,2$, and that all other quantities vanish at infinity. Finally, one obtains

$$iA_1\frac{\partial}{\partial\tau}g_1 + B_1\frac{\partial^2}{\partial\xi^2}g_1 + C_1 g_1|g_1|^2 + F_1^1 g_1\rho_1^2 + F_1^2 g_1\rho_2^2 = 0 \ , \tag{15}$$

$$iA_2\frac{\partial}{\partial\tau}g_2 + B_2\frac{\partial^2}{\partial\xi^2}g_2 + C_2 g_2|g_2|^2 + F_2^1 g_2\rho_1^2 + F_2^2 g_2\rho_2^2 = 0 \ , \tag{16}$$

where A_j, B_j, C_j, and F_j^k $(j,k=1,2)$ are real, known constants. We recall that ρ_1 and ρ_2 are the limits of the amplitudes of g_1 and g_2 as $\xi \to -\infty$.

Equations giving the evolution of the second-order terms are also obtained. Presumably the terms $\vec{H}^2_{1,0}$ and $\vec{M}^2_{1,0}$ depend on $|g_2|^2$ and $\vec{H}^2_{0,1}$ and $\vec{M}^2_{0,1}$ on $|g_1|^2$. The terms $\vec{H}^2_{0,0}$ and $\vec{M}^2_{0,0}$ represent a third wave (a traveling wave) that propagates with a velocity V_0 different from V_1 and V_2. This third wave interacts with both others, but only through their second-order terms. As mentioned before, the complete description of these features, as well as the detailed derivation of Eqs. (15) and (16) and the computation of the explicit values of coefficients A_j, B_j, C_j, and F_j^k $(j,k=1,2)$, are left for future publication.

C. Nonlinear ferromagnetic Faraday effect in the NLS approximation

Let us recall that in Eqs. (15) and (16), the quantities g_1 and g_2 are functions of (ξ_1,τ) and (ξ_2,τ) respectively, where $\xi_j = \xi - V_j T$, and V_1 and V_2 are the group velocities of the two waves and that these functions are proportional to the amplitude of the waves.

For each j, the quantities

$$\rho_j(\tau) = \lim_{\xi_j \to -\infty} |g_j(\xi_j,\tau)| \tag{17}$$

are the amplitudes far away from the region where the modulation undergoes significant variations. Let

$$g_j(\xi_j,\tau) = g_j'(\xi_j,\tau) \times \exp\left[\frac{i}{A_j}\int_0^\tau [F_j^1\rho_1^2(\tau') + F_j^2\rho_2^2(\tau')]d\tau'\right] \ . \tag{18}$$

Then both Eqs. (15) and (16) reduce to the nonlinear Schrödinger equation

$$iA_j\frac{\partial}{\partial\tau}g_j'+B_j\frac{\partial^2}{\partial\xi^2}g_j'+C_jg_j'|g_j'|^2=0\ . \tag{19}$$

It is known [8] that this equation admits two kinds of solution, depending on the sign of B_jC_j. For $B_jC_j>0$, the incident wave is unstable and bunches into solitons (Benjamin-Feir instability [9]). For $B_jC_j<0$ the incident wave is stable and may be modulated by so-called dark-solitons solutions. The sign of the product B_jC_j is obviously the same as for a single wave and has been extensively studied in [4]. The wave with negative helicity is always stable since the wave with positive helicity may or may not be stable depending on the values of α, ω, and θ. For $\theta=0$ it is always unstable.

The term in which we are mainly interested here is the interaction phase factor

$$\exp\left[i\frac{F_1^2}{A_1}\int_0^\tau\rho_2^2(\tau')d\tau'\right] \tag{20}$$

and the corresponding one for the second wave. Note that if the wave packets are localized, i.e., if $\rho_j=0$ for each j, there is, at this approximation, no interaction at all between the two polarizations. Indeed the exponential factor in Eq. (18) is 1 and the NLS equation (19) describes completely both waves. This proves that, for waves vanishing at infinity, the results obtained in [3,4] are still valid when a second wave is present. For waves that do not vanish at infinity, these results are only modified by multiplication of the phase factor (20).

This phase factor depends on the amplitude $\rho_j(\tau)$ of the wave as $\xi_j\to-\infty$. If $\rho_j(\tau)$ is not constant, because there is no delay in its argument, this seems to represent a propagation at infinite velocity, which would be unphysical. This apparent paradox corresponds simply to the assumptions of our perturbative calculus: ρ_1 and ρ_2 are functions of τ and not of T; this means that they vary very slowly in such a way that they are approximately constant during the propagation time of the wave through the sample of ferromagnet, thus their variation during this propagation time can be neglected.

Corresponding to the exponential factor in Eq. (18), there will be a nonlinear term in the Faraday rotation (the polarization direction is determined by the difference ψ between the phases of both circular polarizations). Let us distinguish the different components of this phase difference by

$$\psi=\psi_l+\psi_{nl}+\psi_{NLS}\ . \tag{21}$$

ψ_l represents the linear Faraday rotation. ψ_{nl} is the term coming from the above-mentioned exponential factor

$$\psi_{nl}=\left[\frac{F_1^1}{A_1}-\frac{F_2^1}{A_2}\right]\int_0^\tau[\rho_1(\tau')]^2d\tau' + \left[\frac{F_1^2}{A_1}-\frac{F_2^2}{A_2}\right]\int_0^\tau[\rho_2(\tau')]^2d\tau'\ . \tag{22}$$

For various reasons, in particular because it is difficult to express ψ_{nl} as a function of the variable x in the laboratory (however, ψ_{nl} will always depend on time), but also because expressions of coefficients F_j^k $(j,k=1,2)$ are too complicated, expression (22) is not completely appropriate to compute an expression of the Faraday rotation, which could be experimentally verified. Furthermore, the phase of the solutions of NLS equation (19) is not constant and has a presumably non-negligible effect on Faraday rotation. This is the term that we have called ψ_{NLS} in (21). We will make no attempt to calculate it because this term depends strongly on the form of each solution and even in the very particular case of the simplest solitonic solutions for both g_1' and g_2', the calculus of it is far from being easy.

We have shown that the results obtained for a single wave are still valid and that a nonlinear Faraday rotation exists at the time, space, and amplitude for which the modulations is governed by the NLS equation. The following section is devoted to the computation of the angle of Faraday rotation through an adapted perturbative expansion.

III. PERTURBATIVE METHOD VALID FOR HIGH FREQUENCIES

A. Choice of an adapted perturbative method

Let us denote by u_1 and u_2 the phase velocities and by V_1 and V_2 the group velocities of the two waves of different polarization and the same pulsation ω_0 propagating in a ferromagnetic dielectric. The time scale at which the linear Faraday effect occurs is proportional to the difference u_2-u_1. The nonlinear effect will concern the modulations and the time scale at which it occurs will rather be proportional to V_2-V_1. To perform a perturbative expansion that describes the Faraday effect, this time scale should be very small. Thus it is convenient to seek a limit where the difference u_2-u_1 or V_2-V_1 tends to zero. It is the case for very strong fields, but the symmetry between the two waves is then destroyed. On the other hand, in the limit of high frequencies, both u_2-u_1 and V_2-V_1 tend to zero and the symmetry is retained. Then the pulsation of both waves is chosen as

$$\omega_0=\frac{\omega}{\varepsilon}\ , \tag{23}$$

where ε will be our perturbative parameter ($\varepsilon\ll 1$).

Let us expand in power series of ε the solutions $k_P(\omega)$ and $k_N(\omega)$ of the dispersion relation (13), corresponding to waves with, respectively, positive and negative helicity. Deriving and then inverting this power series, one finds the expansion of the corresponding velocities V_P and V_N,

$$V_P=1-\varepsilon^2v-\varepsilon^3w+O(\varepsilon^4)\ , \tag{24a}$$

$$V_N=1-\varepsilon^2v+\varepsilon^3w+O(\varepsilon^4)\ , \tag{24b}$$

where v and w have the same values as in formulas (30) and (34) below. We see that V_N-V_P is of order ε^3. Let us call ξ and τ the slow variables corresponding to the space and time scales of the searched phenomenon. Then the ratio ξ/τ must be of order ε^3. But the terms of order ε^2 are not zero in the expansion of V_P and V_N, although they cancel in the difference $V_P - V_N$. Hence a dependence of the complex amplitudes of the waves can be expected on a time scale T such that ξ/T is of order ε^2.

It remains to find the order of magnitude of ξ. After some attempts, the best choice is found to be the following: the space and time scales ξ and T of this problem shall correspond to the space and time scales ξ and τ of the NLS approximation. Thus ξ will be of order ε^p and T of order ε^{2p}. The ratio ξ/T is then of order ε^2 if $p=2$. In summary, we define the slow variables as

$$\xi=\varepsilon^2(x-Vt),\quad T=\varepsilon^4 t,\quad \tau=\varepsilon^5 t\ , \tag{25a}$$

that is,

$$\frac{\partial}{\partial x}=\varepsilon^2\frac{\partial}{\partial\xi},\quad \frac{\partial}{\partial t}=-\varepsilon^2 V\frac{\partial}{\partial\xi}+\varepsilon^4\frac{\partial}{\partial T}+\varepsilon^5\frac{\partial}{\partial\tau}\ . \tag{25b}$$

The expansion of the fields $\vec{H}$ and $\vec{M}$ in the harmonics of plane waves $e^{i\varphi_j}$ defined by Eqs. (3)–(5) is still assumed to be valid. The quantities $\vec{M}_n$ and $\vec{H}_n$ will be assumed here to be functions of (ξ,T,τ) and expanded in a power series of ε as in (6). The present computation actually marries a multiscale expansion analogous to the one in Sec. II and an expansion in a powers series of $1/\omega_j$ of all quantities involved. Peculiarly, the wave numbers k_j have to be expanded in a power series of $\varepsilon \propto 1/\omega_j$,

$$k_j=\frac{k_j^0}{\varepsilon}+k_j^1+\varepsilon k_j^2+\cdots\ . \tag{26}$$

The quantities k_j^0 will be calculated order by order during the perturbative calculus. We will obtain the same values as the coefficients of the expansions in power series of ε of the solutions $k_1(\omega)$ and $k_2(\omega)$ of the dispersion relation (13).

As before, the terms $\vec{H}^0_{0,0}$ and $\vec{M}^0_{0,0}$ of the expansion of $\vec{H}$ and $\vec{M}$ will be assumed to be constant. The terms describing the dominant behavior of the amplitude of the two incoming waves should be *a priori* of order ε^2, owing to their order of magnitude in the NLS approximation. But if we replace ω by ω/ε in the expressions of $\vec{h}^1_{1_j}$ and $\vec{m}^1_{1_j}$ of Eq. (12) (the expressions are the same as for a single wave; see [3,4]), we find that

$$\vec{h}^1_{1_j}\simeq m_x\begin{bmatrix}\pm i\varepsilon\dfrac{m_t}{\omega}\\ -i\\ \pm 1\end{bmatrix}, \tag{27a}$$

$$\vec{m}^1_{1_j}\simeq\varepsilon\frac{m_x}{\omega}\begin{bmatrix}\mp i m_t\\ \pm i m_x\\ -m_x\end{bmatrix}. \tag{27b}$$

We use the notation

$$1_j=\begin{cases}(1,0) & \text{for } j=1\\ (0,1) & \text{for } j=2\ .\end{cases}$$

One could think that the choice corresponding to the best analogy to the NLS approximation is the following: the magnetization vector of the jth wave is represented by $\vec{M}^3_{1_j}$ and the corresponding field by $\vec{H}^2_{1_j}$. In fact, we may redefine $\vec{h}^1_{1_j}$, $\vec{m}^1_{1_j}$, and g_j so that $\vec{h}^1_{1_j}$ and $\vec{m}^1_{1_j}$ will be multipled by ω_j and g_j divided by the same quantity. This change is of no significance for the conclusions in the NLS approximation and allows us to represent the magnetization vector of the jth wave by $\vec{M}^2_{1_j}$ and the corresponding field by $\vec{H}^1_{1_j}$. This has the advantage of avoiding amplitudes that are too small while keeping a scaling consistent with the NLS frame.

B. Calculus

We use the stretched coordinates and expansions defined in the preceding subsection in the basic equations (rescaled) (1′) and (2′) and solve them order by order. The calculus does not present any real mathematical difficulty, but is awfully long and complicated. Its structure is described in the Appendix.

The first condition we obtain, apart from those that fix the k_j^n, gives the velocity V, $V=1$. A second condition gives the T dependence of the amplitudes g_1 and g_2 of the two waves. We set

$$\zeta=\xi+vT\ . \tag{28}$$

Then

$$g_1(\xi,T,\tau)=h_1(\zeta,\tau)\exp\left[-i\eta\frac{(2m_x^2+m_t^2)}{2\omega^2 m_x}\times(|h_1|^2-\rho_1^2-\rho_2^2)T\right], \tag{29a}$$

$$g_2(\xi,T,\tau)=h_2(\zeta,\tau)\exp\left[+i\eta\frac{(2m_x^2+m_t^2)}{2\omega^2 m_x}\times(|h_2|^2-\rho_1^2-\rho_2^2)T\right], \tag{29b}$$

where h_1 and h_2 are functions of two variables only ($\zeta=\xi+vT$ and τ), $\eta=\pm1$ and

$$v=\frac{1}{8\omega^2}[(4\alpha+1)m_x^2+2(1+\alpha)m_t^2] \tag{30}$$

is the coefficient of the first nonzero term following $V=1$ in the expansion of the group velocities in power series of $\varepsilon\propto 1/\omega_0$. This first correction to the propagation velocity is the same for both waves. For $j=1,2$, the quantities ρ_j are defined by

$$\rho_j=\lim_{\xi\to-\infty}|g_j|\ , \tag{31}$$

as before. At first order in $\varepsilon\propto 1/\omega_0$, wave 1 has a positive helicity and wave 2 a negative one if we choose $\eta=-1$. We make this choice thereafter.

After a long computation, a third condition gives the τ

evolution of g_1 and g_2 (or h_1 and h_2) and the T evolution of the second-order terms f_1 and f_2 of the amplitudes of the fundamental. It reads, for $j=1,2$,

$$\mathcal{E}_j^g g_j+\mathcal{E}_j^{f,\xi}\frac{\partial f_j}{\partial\xi}+\mathcal{E}_j^{\xi}\frac{\partial g_j}{\partial\xi}+\mathcal{E}_j^{(1)}g_j|g_1|^2+\mathcal{E}_j^{(2)}g_j|g_2|^2+\mathcal{E}_j^{\rho,1}g_j\rho_1^2+\mathcal{E}_j^{\rho,2}g_j\rho_2^2+\mathcal{E}_j^{f,|g|}f_j|g_j|^2+\mathcal{E}_j^{f,\rho}f_j(\rho_1^2+\rho_2^2)+\mathcal{E}_j^{f,*}f_j^*g_j^2+\mathcal{E}_j^{\tau}\frac{\partial g_j}{\partial\tau}+\mathcal{E}_j^{T}\frac{\partial f_j}{\partial T}=0\ , \quad (32)$$

where $\mathcal{E}_j^g$, $\mathcal{E}_j^{f,\xi}$, $\mathcal{E}_j^{\xi}$, $\mathcal{E}_j^{(1)}$, $\mathcal{E}_j^{(2)}$, $\mathcal{E}_j^{\rho,1}$, $\mathcal{E}_j^{\rho,2}$, $\mathcal{E}_j^{f,|g|}$, $\mathcal{E}_j^{f,\rho}$, $\mathcal{E}_j^{f,*}$, $\mathcal{E}_j^{\tau}$, and $\mathcal{E}_j^{T}$ are constant coefficients, explicitly known. There is obviously an analogous equation for the second wave.

Equation (32) can be separated into two equations, one of which gives the T evolution of f_j, the second giving the τ evolution h_j. The latter equation can be solved by means of changes of variables (see the Appendix). The result reads as follows: we call α_1 and α_2 the respective phases of h_1 and h_2 and introduce the variables

$$X=\zeta-w\tau,\quad Y=\zeta+w\tau\ , \quad (33)$$

where

$$w=\frac{1}{8\omega^3 m_x}[(8\alpha^2+4\alpha+1)m_x^4+2(4\alpha+1)m_x^2(1+\alpha)m_t^2+(1+\alpha)^2m_t^4] \quad (34)$$

is the correction to the group velocity at this order. More precisely, in Eq. (33), $\pm w$ is the third nonzero coefficient, after 1 and v, in the expansion of the group velocity in a power series of $\varepsilon\propto 1/\omega$. Then

$$h_1=a_1(Y)e^{i\alpha_1(Y,\tau)}\ , \quad (35)$$

where a_1 is an arbitrary real function and

$$\alpha_1=\frac{-1}{2\eta w}\left[\{A[a_1(Y)]^2-D\rho_1^2-E\rho_2^2\}X+B\int_{X_0(Y)}^{X}[a_2(X')]^2dX'+\eta Ca_1(Y)\int_{X_1(Y)}^{X}2\,\mathrm{Re}[u_1(X',Y)]dX'\right]+\alpha_1^0(Y)\ . \quad (36)$$

The real constants A, B, C, D, and E are explicitly known [see (A17) and [10]]. In a similar way,

$$h_2=a_2(X)e^{i\alpha_2(X,\tau)}\ , \quad (37)$$

where a_2 is an arbitrary real function and

$$\alpha_2=\frac{1}{2\eta w}\left[\{A[a_2(X)]^2-D\rho_2^2-E\rho_1^2\}Y+B\int_{Y_0(X)}^{Y}[a_1(Y')]^2dY'-\eta Ca_2(X)\int_{Y_1(X)}^{Y}2\,\mathrm{Re}[u_2(X,Y')]dY'\right]+\alpha_2^0(X)\ . \quad (38)$$

The functions $X_0(Y)$, $X_1(Y)$, $\alpha_1^0(Y)$, $Y_0(X)$, $Y_1(X)$, and $\alpha_2^0(X)$ are arbitrary "integration constants."

C. Calculus of the angle of nonlinear Faraday rotation

1. Generalized Stokes parameters

Let us consider two waves propagating in the same medium with the same amplitude and different circular polarizations. Let us call k_+ (k_-) the wave number of the positive (negative) helicity. One can show (see [5], p. 119 and following) that after a propagation on the distance x, the linear polarization of the sum of the two waves has rotated for an angle χx, where $\chi=\frac{1}{2}(k_+-k_-)$ is the angle of Faraday rotation by a unit of length.

If the amplitudes differ, the polarization is elliptic and is characterized by the Stokes parameters. These parameters are usually defined (see [11], p. 30) in relation to the amplitudes of the two linear polarizations that make up the wave. We recompute them in relation to the complex amplitudes A and B of both circular polarizations that make up the wave: if $\mathcal{E}_+$ ($\mathcal{E}_-$) is the wave with positive (negative) helicity,

$$\mathcal{E}_+=\mathrm{Re}\left[E_+\begin{bmatrix}0\\1\\+i\end{bmatrix}e^{i(kx-\omega t)}\right]\ , \quad (39a)$$

$$\mathcal{E}_-=\mathrm{Re}\left[E_-\begin{bmatrix}0\\1\\-i\end{bmatrix}e^{i(kx-\omega t)}\right]\ , \quad (39b)$$

then the Stokes parameters are

$$s_0=2(|E_+|^2+|E_-|^2)\ , \quad (40a)$$

$$s_1=4\,\mathrm{Re}(E_+E_-^*)\ , \quad (40b)$$

$$s_2=4\,\mathrm{Im}(E_+E_-^*)\ , \quad (40c)$$

$$s_3=2(|E_-|^2-|E_+|^2)\ . \quad (40d)$$

One sees that s_1 and s_2 are simply related to the phase difference between the two waves. By the Faraday effect, this difference increases by an amount $2\chi x$ after a propagation on a distance x. Thus angle χ is directly related to the parameters of the waves that may be determined from experiment, even if the complete wave is not linearly polarized. The two waves under consideration here are circularly polarized only at the first nonzero order in $\varepsilon\propto 1/\omega_0$. We will neglect this fact and define the generalized Stokes parameters using expressions (40). Note also that Stokes parameters are usually defined in relation to the electric field. Since we are studying magnetic waves here, we will define the generalized Stokes parameters in relation to the magnetic field $\vec{H}$.

Thus we define the parameters

$$\chi=\frac{1}{2}[k_1-k_2]+\frac{1}{2x}[\arg(g_1)-\arg(g_2)] \;, \tag{41}$$

$$s_0=\tfrac{1}{2}(\|\vec{H}_1\|^2_{\text{eff}}+\|\vec{H}_2\|^2_{\text{eff}}) \;, \tag{42a}$$

$$s_1=-\|\vec{H}_1\|^2_{\text{eff}}\|\vec{H}_2\|^2_{\text{eff}}\cos(2\chi x) \;, \tag{42b}$$

$$s_2=\|\vec{H}_1\|^2_{\text{eff}}\|\vec{H}_2\|^2_{\text{eff}}\sin(2\chi x) \;, \tag{42c}$$

$$s_3=-\tfrac{1}{2}(\|\vec{H}_1\|^2_{\text{eff}}-\|\vec{H}_2\|^2_{\text{eff}}) \;. \tag{42d}$$

Here $\vec{H}_1$ and $\vec{H}_2$ are the complex vectorial amplitudes of the magnetic field of both waves

$$\vec{H}_j=\varepsilon\vec{H}^1_{1j}+\varepsilon^2\vec{H}^2_{1j}+O(\varepsilon^3) \tag{43}$$

and the effective norm $\|\;\|_{\text{eff}}$, defined by

$$\|\vec{u}\|^2_{\text{eff}}=2\vec{u}\cdot\vec{u}^* \;, \tag{44}$$

is such that $\|\vec{H}_j\|_{\text{eff}}$ is the mean value of the norm of the amplitude of the wave j in a duration large in regard to the period of the carrier wave but small in regard to the time scale of the variation of its modulation.

2. Expression of the rotation angle

χ breaks up in the following way:

$$\chi=\chi_l+\chi_m+\chi_{\text{nl}n}+\chi_{\text{nl}a} \tag{45}$$

χ_l is the linear part of χ, χ_m is a term that depends only on arbitrary phase modulations of both waves that propagate with their own group velocities, $\chi_{\text{nl}n}$ is the angle corresponding to the normal nonlinear Faraday effect, and $\chi_{\text{nl}a}$ is the angle corresponding to anomalous nonlinear Faraday effect. $\chi_l=\frac{1}{2}(k_1-k_2)$ has been calculated up to order ε^4,

$$\begin{aligned}\chi_l=&-\frac{m\cos\theta}{2}-\frac{m^3}{16\omega_0^2\cos\theta}[\alpha(\alpha-4)\cos^4\theta+6\alpha(\alpha+1)\cos^2\theta+(1+\alpha)^2]\\&-\frac{m^5}{256\omega_0^4\cos^3\theta}[(128\alpha^4+128\alpha^3+96\alpha^2+40\alpha+7)\cos^8\theta\\&\qquad+4(1+\alpha)(64\alpha^3+48\alpha^2+24\alpha+5)\cos^6\theta\sin^2\theta+18(1+\alpha)^2(8\alpha^2+4\alpha+1)\cos^4\theta\sin^4\theta\\&\qquad+4(1+\alpha)^3(4\alpha+1)\cos^2\theta\sin^6\theta-(1+\alpha)^4\sin^8\theta]+O\left[\frac{1}{\omega_0^6}\right] \;.\end{aligned} \tag{46}$$

(We recall that $\omega_0=\omega/\varepsilon$.)

The calculus of the nonlinear terms necessitates some work in order to come back into the laboratory. Using definitions of all the variables used (25), (28), and (33), we obtain

$$X=\varepsilon^2(x-V_2t) \;, \tag{47a}$$

$$Y=\varepsilon^2(x-V_1t) \;, \tag{47b}$$

with

$$V_1=V-\varepsilon^2v-\varepsilon^3w \;, \tag{48a}$$

$$V_2=V-\varepsilon^2v+\varepsilon^3w \;. \tag{48b}$$

Then we rescale the coefficients appearing in Eqs. (36) and (38). Furthermore, a_1 and a_2 and their limits ρ_1 and ρ_2 come from the term of order ε^1 of the field, thus the corresponding quantities measured in the laboratory are

$$\hat{a}_1=\varepsilon a_1,\;\; \hat{a}_2=\varepsilon a_2,\;\; \hat{\rho}_1=\varepsilon\rho_1,\;\; \hat{\rho}_1=\varepsilon\rho_1 \;, \tag{49}$$

and so on (more detail is given in the Appendix). Thus we obtain expressions for the phases as functions of the quantities measured in the laboratory.

α_1 is defined apart from an arbitrary function of Y, i.e., of $x-V_1t$, and α_2 is defined apart from an arbitrary function of X, i.e., of $x-V_2t$. These arbitrary functions can be modified in order to express (as much as possible) the significant parts of the phases as functions of x only. To this aim, we use the identities

$$t=\frac{x}{V_1}-\frac{1}{V_1}(x-V_1t) \;, \tag{50}$$

$$x-V_2t=\frac{-2\hat{w}}{V_1}x+\frac{V_2}{V_1}(x-V_1t) \;, \tag{51}$$

and the analog for $x-V_1t$. We compute or simplify, in an analogous way, the quadratures involved by Eqs. (36) and (38). So we obtain an adequate expression for the phases of g_1 and g_2 (A38). Finally, we get

$$\tfrac{1}{2}[\arg(g_1)-\arg(g_2)]=x(\chi_m+\chi_{\text{nl}n}+\chi_{\text{nl}a}) \;, \tag{52}$$

where

$$x\chi_m=\tfrac{1}{2}[\alpha_1^{0\prime}(x-V_1t)-\alpha_2^{0\prime}(x-V_2t)] \;, \tag{53}$$

$$\chi_{\text{nl}n}=\frac{m}{\omega_0^2}\frac{1+\cos^2\theta}{16\cos\theta}[\|\vec{H}_1\|^2_{\text{eff}}+\|\vec{H}_2\|^2_{\text{eff}}-2(\|\vec{H}_1\|^2_\infty+\|\vec{H}_2\|^2_\infty)] \;, \tag{54}$$

$$\chi_{\mathrm{nl}a}=\frac{m^2}{\omega_0^3}\left\{P(\|\vec{H}_1\|_\infty^2-\|\vec{H}_2\|_\infty^2)+Q(\|\vec{H}_1\|_{\mathrm{eff}}^2-\|\vec{H}_2\|_{\mathrm{eff}}^2)\right.$$

$$\left.+R\frac{1}{x}\int_0^x\left[\|\vec{H}_1\|_{\mathrm{eff}}^2\Big|_{2\hat{\omega}/V_2(x'-x)+x-V_1t}-\|\vec{H}_2\|_{\mathrm{eff}}^2\Big|_{-2\hat{\omega}/V_1(x'-x)+x-V_2t}\right]dx'\right\} \tag{55}$$

with

$$P=\frac{-1}{16\cos^2\theta}[3\alpha\sin^4\theta+3(1-4\alpha)\sin^2\theta+8\alpha]\ , \tag{56a}$$

$$Q=\frac{1}{64\cos^2\theta}[(7\alpha-13)\sin^4\theta+16(1-2\alpha)\sin^2\theta+4(6\alpha-1)]\ , \tag{56b}$$

$$R=\frac{1}{16\alpha}[4\alpha(1+\alpha)\cos^2\theta+(3\alpha^2+2\alpha-4)\sin^2\theta]+\frac{(1+\alpha)\sin^4\theta}{8\alpha(\alpha+\sin^2\theta)}\ . \tag{56c}$$

D. Normal and anomalous ferromagnetic Faraday effects

The $x\chi_m$ term corresponds to the phase difference between two wave envelopes of arbitrary shape, each propagating with its own group velocity V_1 or V_2. The corresponding Faraday rotation angle depends completely on the shape of the phase modulation of both wave packets; it is a linear effect of these wave modulations. This term can be easily computed if the phase modulations are precisely known, which would allow measurements of the nonlinear terms. It disappears when the input waves are not modulated in phase.

Using definition (42) of the generalized Stokes parameters and identifying by the subscript ∞ their values as $\xi\to-\infty$, one can write $\chi_{\mathrm{nl}n}$ and $\chi_{\mathrm{nl}a}$ in terms of these parameters

$$\chi_{\mathrm{nl}n}=\frac{m}{\omega_0^2}\frac{1+\cos^2\theta}{8\cos\theta}[s_0-2s_{0,\infty}]\ , \tag{57}$$

$$\chi_{\mathrm{nl}a}=\frac{-2m^2}{\omega_0^3}[P\,s_{3,\infty}+Q\,s_3+R\,\langle s_3\rangle]\ , \tag{58}$$

where

$$\langle s_3\rangle=\frac{1}{x}\int_0^x\frac{-1}{2}\left[\|\vec{H}_1\|_{\mathrm{eff}}^2\Big|_{2\hat{\omega}/V_2(x'-x)+x-V_1t}-\|\vec{H}_2\|_{\mathrm{eff}}^2\Big|_{-2\hat{\omega}/V_1(x'-x)}+x-V_2t\right]dx'\ . \tag{59}$$

Although it is not a functional of s_3 from the mathematical point of view, $\langle s_3\rangle$ can be interpreted in a certain sense as a mean value of s_3. This justifies our notation.

$\chi_{\mathrm{nl}n}$ is the angle of nonlinear normal Faraday rotation: if the wave is reflected, the exterior field $\vec{H}_0$ is seen from the wave so that θ becomes $\pi-\theta$, thus $\chi_{\mathrm{nl}n}$ changes its sign and the rotation is thus the same as before. Unlike the latter, $\chi_{\mathrm{nl}a}$ is not affected by the change of θ into $\pi-\theta$ [see formulas (56)]. Thus the corresponding rotation is canceled and not doubled by reflection: that is why we call this effect "anomalous."

It is possible to cancel the normal effect while multiplying the anomalous one. If the wave runs through a first sample of length x, the exterior field $\vec{H}_0$ making an angle θ with the propagation direction, and then through a second sample identical to the first one, with an exterior field of same magnitude, but making an angle $\pi-\theta$ with the propagation direction, the normal Faraday rotation will be canceled and the anomalous one be doubled. We can expect to obtain the same result with an exterior field varying very slowly in space. Note that the preceding calculus may be valid only if the length scale of this variation is very large in regard to the length scale of the variation of the wave modulation, which must be itself very large in regard to the wavelength of the carrier.

Observe that $\chi_{\mathrm{nl}n}$ is proportional to the parameter s_0 that measures the energy density of the waves. In the case of a short pulse, $s_{0,\infty}=0$, thus

$$\chi_{\mathrm{nl}n}=\frac{m}{\omega_0^2}\frac{1+\cos^2\theta}{8\cos\theta}s_0\ , \tag{60}$$

while for a plane wave $s_{0,\infty}=s_0$, thus

$$\chi_{\mathrm{nl}n}=-\frac{m}{\omega_0^2}\frac{1+\cos^2\theta}{8\cos\theta}s_0\ . \tag{61}$$

Notice that the angle of normal nonlinear Faraday rotation for a plane wave is opposite that for a short pulse. $\chi_{\mathrm{nl}a}$ depends on s_3, which is the difference between the energy of both elliptic polarizations. Thus it is zero for a linear incident polarization and would be nonzero for an elliptic one.

$\chi_{\mathrm{nl}n}$ and even more $\chi_{\mathrm{nl}a}$ are *a priori* very small quantities: they were of order ε^4 and ε^5, respectively, but the order of magnitude can be increased by different ways. First $\chi_{\mathrm{nl}n}$ is proportional to $1/\cos\theta$ and the coefficients P and Q in $\chi_{\mathrm{nl}a}$ to $1/\cos^2\theta$, thus as θ approaches $\pi/2$, they tend to infinity (note that for $\theta=\pi/2$ the preceding calculus is not valid). Let us give asymptotic values of the nonlinear rotation angles as $\theta\to\pi/2$:

$$\chi_{\mathrm{nl}n}\simeq\frac{m}{\omega_0^2}\frac{1}{8\cos\theta}(s_0-2s_{0,\infty})\ , \tag{62}$$

$$\chi_{\mathrm{nl}a}\simeq\frac{-2m^2}{\omega_0^3}[Ps_{3,\infty}+Qs_3]\ , \tag{63}$$

$$P \simeq \frac{\alpha-3}{16\cos^2\theta} , \tag{64a}$$

$$Q \simeq \frac{-(1+\alpha)}{64\cos^2\theta} . \tag{64b}$$

It is also possible to increase the anomalous nonlinear rotation by use of a strong exterior field; the preceding coefficients tends to infinity as α, which measures the strength of the exterior field $\vec{H}_0$. Furthermore, the wave fields $\vec{H}_1$ and $\vec{H}_2$ are assumed to be small, but this smallness is relative to $\vec{H}_0$. Thus if $\|\vec{H}_0\|$ takes very large values, a higher wave power can be used without exiting from the small amplitude approximation. Since $\chi_{\text{nl}n}$ and $\chi_{\text{nl}a}$ are proportional to $\|\vec{H}_1\|^2_{\text{eff}} \pm \|\vec{H}_2\|^2_{\text{eff}}$, this is of great importance. Let us give the asymptotic expressions of the coefficients involved in expression (59) of $\chi_{\text{nl}a}$ for strong exterior fields:

$$P \simeq \frac{-\alpha}{16\cos^2\theta}[3\sin^4\theta - 12\sin^2\theta + 8] , \tag{65a}$$

$$Q \simeq \frac{\alpha}{64\cos^2\theta}[7\sin^4\theta - 32\sin^2\theta + 24] , \tag{65b}$$

$$R \simeq \frac{\alpha}{16}(3+\cos^2\theta) . \tag{65c}$$

The three coefficients P, Q, and R have the same order of magnitude and thus, for a strong exterior field, the anomalous nonlinear Faraday rotation is proportional to the exterior field.

IV. CONCLUSION

In Sec. II we showed that the evolution of the modulation of the two waves with the same frequency and opposite polarization that may propagate together in a ferromagnetic dielectric is described by two independent NLS equations and that the main behavior of the solutions of these equations is not affected by any interaction of both waves. An interaction term has been found, depending on the value of the waves at infinity and affecting only the phase of the waves. This term is a nonlinear contribution to the Faraday rotation. The phase of the solution of the NLS equation contributes also to this effect. Hence we have showed the existence of a nonlinear Faraday effect at the time, space, and amplitude scales where the propagation is described by the NLS equation.

However, this calculus does not give a convenient expression of the nonlinear Faraday effect.

In Sec. III we computed the angle of nonlinear Faraday rotation with an adequate perturbation scheme. We found both normal and anomalous nonlinear Faraday effects. Both have been expressed in terms of generalized Stokes parameters of the wave. Then we have discussed different ways to make evident the various computed terms, assuming a quasitransverse, or strong, or slowly space varying, exterior magnetic field.

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APPENDIX

In this appendix we give some precision on the computation involved in Sec. III of this paper. We shall record only the structure of the calculus. All other details and the values of the quantities involved in it can be found in [10] (see Chap. V, Appendix).

1. Derivation of the equations

The basic system (1′) and (2′) has been given in Sec. II A and the perturbative method we use is described in Sec. III A. Let us start with the order by order resolution.

The first step, at order ε^{-1} in Eq. (1′) and ε^0 in Eq. (2′), gives the terms of order 0, which are the same as for the NLS approximation, given by (10) and (11). It gives also the value of $k_1^0 = k_2^0 = \omega$ and shows that $\vec{M}^1_{1_j} = 0$ and $H^{1,x}_{1_j} = 0$. All other terms of order 1 are also taken to be zero except $H^{1,y}_{1_j}$ and $H^{1,z}_{1_j}$, which are left free by the equations of this order.

The second step, order ε^0 in Eq. (1′) and ε^1 in Eq. (2′), gives the values of k_j^1,

$$k_1^1 = -k_2^1 = \frac{\eta m_x}{2} \tag{A1}$$

with $\eta = \pm 1$, and the values of $H^{1,s}_{1_j}$ for $j=1,2$ and $s=y,z$. This completes the results at order 1:

$$\vec{H}^1_{1,0} = \vec{h}^1_{1,0}\, g_1 , \quad \vec{H}^1_{0,1} = \vec{h}^1_{0,1}\, g_2 , \tag{A2}$$

$$\vec{h}^1_{1,0} = \begin{bmatrix} 0 \\ i\eta \\ 1 \end{bmatrix} , \quad \vec{h}^1_{0,1} = \begin{bmatrix} 0 \\ -i\eta \\ 1 \end{bmatrix} , \tag{A3}$$

where g_1 and g_2 are unknown functions of (ξ, T, τ). As written above, all other terms are zero at this order. The polarization vectors $\vec{h}^1_{1,0}$ and $\vec{h}^1_{0,1}$ correspond to the two elliptic (circular in this first approximation) polarizations. $\vec{h}^1_{1,0}$ corresponds to a left-handed or positive polarization and $\vec{h}^1_{0,1}$ to a right-handed or negative one, if we choose $\eta = -1$. The equations at this order give also the values of all the results for the terms of order 2, except $H^{2,s}_{1_j}$, $s=y,z$, and the terms corresponding to $n=(0,0)$ and $n=(1,-1)$ for which the vanishing of many coefficients necessitates a search of the equations at a higher order. This structure will repeat at each step of the calculus. For convenience, we will give the results put together order by order.

Thus at order 2, one obtains

$$\vec{M}^2_{1,0} = \vec{m}^2_{1,0}\, g_1 , \tag{A4a}$$

$$\vec{H}^2_{1,0} = \vec{h}^1_{1,0}\, f_1 + \vec{h}^2_{1,0}\, g_1 , \tag{A4b}$$

where $m^2_{1,0}$ and $h^2_{1,0}$ are constant vectors (given in [10]). f_1 is an unknown function of (ξ, T, τ) and its coefficient is the same as that of g_1 in (A2). There are analogous formulas for the (0,1) terms: all formulas concerning (n_2, n_1) are obtained from those that concern (n_1, n_2) by inverting indices 1 and 2 for g_j, f_j, etc., and replacing η by $-\eta$. For this reason, we will not replace η by its value -1 before the end of the computation. Other harmonics are zero at this order. One obtains the values of k_1^2 and k_2^2 too.

At order 3, one obtains a condition that gives the value of the velocity V,

$$V=1\ . \tag{A5}$$

The following terms k_1^3 and k_2^3 of the dispersion relation are obtained too. Furthermore,

$$\vec{M}^3_{1,0}=\vec{m}^2_{1,0}\, f_1+\vec{m}^3_{1,0}\, g_1\ , \tag{A6a}$$

$$\vec{H}^3_{1,0}=\vec{h}^1_{1,0}\varphi_1+\vec{h}^2_{1,0}\, f_1+\vec{h}^3_{1,0}\, g_1\ . \tag{A6b}$$

A third unknown function φ_1 must be introduced; its coefficient is that of f_1 in $\vec{H}^2_{1,0}$ and so on. Nonzero harmonics do appear at this order in the expression of the magnetic field: their general expression is typically of the form

$$\vec{M}^p_{n_1,n_2}=\vec{m}^p_{n_1,n_2}g_1^{n_1}g_2^{n_2}\ , \tag{A7}$$

where we have set

$$g_j^{n_j}=(g_j^*)^{|n_j|}\quad \text{if } n_j<0 \quad \text{for } j=1,2\ . \tag{A8}$$

They are obtained for $n=(2,0)$ and $(1,1)$ and their symmetrics (by permutation of both waves and by complex conjugation).

For $n=(0,0)$ and $(1,-1)$, the vanishing of many coefficients necessitates a search for conditions at a high order; thus much more terms are involved in the expressions, which increases the size of the algebraic part of the work. On the other hand, we obtain not only algebraic equations but also differential equations, which are very easy to integrate but may present questions about boundary conditions.

At this order 3, there is no particular problem; the only nonzero one of these terms is

$$\vec{H}^3_0=\frac{2\eta m_t}{\omega m_x}(|g_1|^2-|g_2|^2)\begin{bmatrix}0\\1\\0\end{bmatrix}\ . \tag{A9}$$

At order 4, we obtain the terms k_1^4, k_2^4,

$$\vec{M}^4_{1,0}=\vec{m}^2_{1,0}\varphi_1+\vec{m}^3_{1,0}\, f_1+\vec{m}^4_{1,0}\, g_1\ , \tag{A10}$$

$$\vec{H}^4_{1,0}=\vec{h}^1_{1,0}\psi_1+\vec{h}^2_{1,0}\varphi_1+\vec{h}^3_{1,0}\, f_1+\vec{h}^4_{1,0}\, g_1\ , \tag{A11}$$

(ψ_1 is the unknown function introduced at this order), and nonzero harmonics $\vec{M}^4_n$ and $\vec{H}^4_n$ for $n=(0,0)$, $(2,0)$, $(1,1)$, $(1,-1)$, $(2,1)$, and the symmetric terms. The solvability condition for $n=(1,0)$ and $(0,1)$ at fifth order gives the T dependence of g_1 and g_2 as

$$\frac{\partial g_1}{\partial T}-v\frac{\partial g_1}{\partial \xi}=-i\eta\frac{(2\omega_x^2+m_t^2)}{2\omega^2 m_x}g_1(|g_1|^2-(\rho_1^2+\rho_2^2))\ , \tag{A12a}$$

$$\frac{\partial g_2}{\partial T}-v\frac{\partial g_2}{\partial \xi}=+i\eta\frac{(2m_x^2+m_t^2)}{2\omega^2 m_x}g_2(|g_2|^2-(\rho_1^2+\rho_2^2))\ , \tag{A12b}$$

where v is defined by (30) and ρ_1 and ρ_2 by (31).

The solutions of Eqs. (A12a) and (A12b) are given by expressions (29a) and (29b). It is convenient to pursue the calculus in terms of the functions g_1 and g_2, using Eqs. (A12a) and (A12b), and not to replace g_1 and g_2 by their expressions (29a) and (29b) before the equations giving the τ dependence of g_1 and g_2 have been obtained.

At order 5, one obtains

$$\vec{M}^5_{1,0}=\vec{m}^2_{1,0}\psi_1+\vec{m}^3_{1,0}\varphi_1+\vec{m}^4_{1,0}\, f_1+\vec{m}^5_{1,0}\, g_1+\vec{m}^{5,\xi}_{1,0}\frac{\partial g_1}{\partial \xi}\ , \tag{A13a}$$

$$\vec{H}^5_{1,0}=\vec{h}^1_{1,0}\chi_1+\vec{h}^2_{1,0}\psi_1+\vec{h}^3_{1,0}\varphi_1+\vec{h}^4_{1,0}\, f_1+\vec{h}^5_{1,0}\, g_1 +\vec{h}^{5,(1)}_{1,0}g_1|g_1|^2+\vec{h}^{5,(2)}_{1,0}g_1|g_2|^2+\vec{h}^{5,\xi}_{1,0}\frac{\partial g_1}{\partial \xi} \tag{A13b}$$

(χ_1 is the unknown function introduced at this order). One obtains also the following term k_j^5 of the expansion of k_j and a list of nonzero harmonics: these are found for $n=(0,0)$, $(2,0)$, $(1,1)$, $(1,-1)$, $(2,1)$, and the symmetric terms for $\vec{M}^5_n$ and for the same values of n, but also $(3,0)$ and $(2,-1)$ (plus eventually some nonzero solutions of homogeneous, linear systems that are not completely determined by equations of this order) and the symmetric terms for the harmonics $\vec{H}^5_n$ of the magnetic field.

At this step the calculus has also given the following partial results for the terms of order 6:

$$\vec{M}^6_{1,0}=\vec{m}^2_{1,0}\chi_1+\vec{m}^3_{1,0}\psi_1+\vec{m}^4_{1,0}\varphi_1+\vec{m}^5_{1,0}\, f_1+\vec{m}^6_{1,0}\, g_1 +\vec{m}^{6,(1)}_{1,0}g_1|g_1|^2+\vec{m}^{6(2)}_{1,0}g_1|g_2|^2 +\vec{m}^{6,\rho}_{1,0}\, g_1(\rho_1^2+\rho_2^2)+\vec{m}^{5,\xi}_{1,0}\frac{\partial f_1}{\partial \xi}+\vec{m}^{6,\xi}_{1,0}\frac{\partial g_1}{\partial \xi} \tag{A14}$$

and the harmonics $\vec{M}^6_n$ for $n=(2,0)$, $(1,1)$, $(3,0)$, $(2,1)$, $(2,-1)$, and the symmetric terms. Note that $\vec{H}^5_{2,1}$ and $\vec{M}^5_{2,1}$ are not completely determined at this step.

At the next step [which is in fact the seventh one, at order ε^5 in Eq. (1′) and ε^6 in Eq. (2′)], we write only the solvability condition for $n=(1,0)$ $(j=1)$ and $(0,1)$ $(j=2)$. Coefficients of χ_j, ψ_j, φ_j, f_j, and $\partial\varphi_j/\partial\xi$ $(j=1,2)$ cancel and we obtain Eq. (32).

2. Resolution of the equation governing the phase evolution

The first step of the resolution of Eq. (32) is to replace g_j $(j=1,2)$ by their expressions (29a) and (29b) in terms of functions h_j $(j=1,2)$. We describe the calculus for $j=1$; the case $j=2$ is similar. We see that it is convenient to set

$$f_1(\xi,T,\tau)=q_1(\xi,T,\tau)e^{i\beta_1 T}\ , \tag{A15}$$

where

$$\beta_1=-\eta C(|h_1|^2-\rho_1^2-\rho_2^2) \tag{A16}$$

with

$$C=\frac{(2m_x^2+m_t^2)}{2\omega^2 m_x}\ . \tag{A17}$$

β_1 is a function of $\zeta=\xi+vT$ and τ and is such that $g_1=h_1 e^{i\beta_1 T}$. It is also convenient to use the coordinates

$$\zeta=\xi+vT,\quad T'=T,\quad \tau=\tau\ . \tag{A18}$$

Let

$$\Phi_a^1=\mathcal{E}_1^g h_1+\mathcal{E}_1^\xi\frac{\partial h_1}{\partial\zeta}+\mathcal{E}_1^{(1)}h_1|h_1|^2+\mathcal{E}_1^{(2)}h_1|h_2|^2+\mathcal{E}_1^{\rho,1}h_1\rho_1^2+\mathcal{E}_1^{\rho,2}h_1\rho_2^2+\mathcal{E}_1^\tau\frac{\partial h_1}{\partial\tau}\ , \tag{A19}$$

$$\Phi_b^1=\mathcal{E}_1^{f,\xi}\frac{\partial q_1}{\partial\xi}+\mathcal{E}_1^{f,|g|}q_1|h_1|^2+\mathcal{E}_1^{f,\rho}q_1(\rho_1^2+\rho_2^2)+\mathcal{E}_1^{f,*}q_1^*h_1^2+\mathcal{E}_1^T\left[\frac{\partial q_1}{\partial T}+v\frac{\partial q_1}{\partial\zeta}+i\beta_1 q_1\right]\ , \tag{A20}$$

$$\Xi_1=i\left[\mathcal{E}_1^\xi\frac{\partial\beta_1}{\partial\zeta}+\mathcal{E}_1^\tau\frac{\partial\beta_1}{\partial\tau}\right]h_1\ . \tag{A21}$$

Equation (32) can be written

$$\Phi_a^1+\Phi_b^1+\Xi_1 T'=0\ . \tag{A22}$$

Φ_a^1 and Ξ_1 depend only on ζ and τ, and Φ_b^1 must be bounded as T' becomes infinite. Thus

$$\Xi_1=0\ , \tag{A23a}$$

$$\Phi_a^1+\Phi_b^1=0\ . \tag{A23b}$$

Equation (A23a) is immediately solvable. Let

$$Y=\zeta-\eta w\tau,\quad \tau'=\tau\ . \tag{A24}$$

Then we get expression (35) for h_1, where a_1 and α_1 are *(a priori)* arbitrary real functions.

Some terms in Φ_b^1 cancel and it is convenient to let

$$q_1=u_1 e^{i\alpha_1}\ , \tag{A25}$$

where u_1 is an *a priori* an unknown complex function of (ζ,T',τ). Using the fact that Φ_a^1 is onstant in regard to T', (A23b) becomes an ordinary differential equation for $u_1(T')$, which is easily solved by separating the real and the imaginary part of $u_1(T')$. One can show that the only bounded solution of (A23b) is u_1 constant in regard to T'. Thus f_1 depends on T in the same way as g_1. The only nonzero term remaining in Φ_b^1 is the nonlinear one, proportional to $a_1^2\mathrm{Re}(u_1)$.

The coefficient $\mathcal{E}_1^g$ of g_1 in (32) depends on k_1^6, which is not yet fixed. We determine it by the linear limit of (32), i.e., we assume in (32) that the derivatives and the square and cubic terms are negligible. The obtained equation is

$$\mathcal{E}_1^g g_1=0\ ; \tag{A26}$$

thus $\mathcal{E}_1^g=0$, which determines k_1^6.

After division of Eq. (A23b) by an adequate factor, we obtain

$$\eta w\frac{\partial\alpha_1}{\partial\zeta}+\frac{\partial\alpha_1}{\partial\tau}+Aa_1^2+Ba_2^2-D\rho_1^2-E\rho_2^2+\eta Ca_1(u_1+u_1^*)=0\ , \tag{A27}$$

where A, B, C, D, and E are real constants, explicitly known [see (A17) and [10]]. The equation for the second wave is

$$-\eta w\frac{\partial\alpha_2}{\partial\zeta}+\frac{\partial\alpha_2}{\partial\tau}+Aa_2^2+Ba_1^2-D\rho_2^2-E\rho_1^2-\eta Ca_2(u_2+u_2^*)=0\ . \tag{A28}$$

a_2, α_2, and u_2 are defined similarly to a_1, α_1 and u_1, and a_2 is a function of the single variable

$$X=\zeta+\eta w\tau\ . \tag{A29}$$

Equations (A27) and (A28) are solved easily using the variables X and Y. Then the expressions (36) and (38) of α_1 and α_2 are obtained.

3. Returning to the laboratory frame

At first order in $\varepsilon\propto 1/\omega_0$, wave 1 has a positive helicity and wave 2 a negative one if we choose $\eta=-1$. We make this choice thereafter. It is consistent with our definition (42) of the generalized Stokes parameters. In preceding subsection, we get expressions (36) and (38) for the phases α_1 and α_2. The complete phases $\arg(g_1)$ and $\arg(g_2)$ can then be computed using (29a) and (29b). In order to deduce a convenient expression for the angle χ defined by (41), which measures the polarization direction, we will express these phases as functions of the laboratory variables. Expressions of X, Y, V_1, and V_2 are given by (47) and (48). Then each coefficient appearing in Eqs. (36) and (38) is rewritten as

$$A=\frac{m^2}{\varepsilon^3\omega_0^2}\mathcal{A}\ , \tag{A30}$$

where $\mathcal{A}$ depends on the angle θ and the ratio α only. We write similar expressions for B, D, E, and

$$C=\frac{m}{\varepsilon^2\omega_0^2}\mathcal{C} \tag{A31}$$

where $\mathcal{C}$ depends on θ only.

We write also

$$w=\frac{\hat{w}}{\varepsilon^3} \tag{A32}$$

($\hat{w}$ depends on ω_0, m, θ, and α). Quantities a_1,a_2,ρ_1, and ρ_2 are rescaled as in Eq. (49) and, in a similar way, we define

$$\hat{u}_1=\varepsilon^2u_1,\quad \hat{u}_2=\varepsilon^2u_2\ . \tag{A33}$$

Using formulas (47)–(49) and (A30)–(A33) in the expressions (36) and (38) of α_1 and α_2, we obtain their expressions as functions of the quantities measured in the laboratory. Using (A31), (A33), and the definition (25) of the variable T in the expression (A16) of the phase terms $\beta_1 T$ and $\beta_2 T$, we obtain such expressions for them. Using Eqs. (50) and (51), we want to modify the arbitrary function of Y, i.e., of $x-V_1t$, involved by expression of α_1 and make the similar transformation for the second wave.

However, in Eqs. (36) and (38) there

are some quadratures. Computation of the quantities $\int_{X_1(Y)}^{X} 2\,\mathrm{Re}[u_1(X',Y)]dX'$ and $\int_{Y_1(X)}^{Y} 2\,\mathrm{Re}[u_2(X,Y')]dY'$ necessitates the knowledge of the τ dependence of u_1 and u_2, i.e., of f_1 and f_2. An equation giving this dependence would be found at the next order of the expansion. To avoid a monstrous calculus, we notice that the sequence $g_j, f_j, \varphi_j, \psi_j, \chi_j, \ldots$ (for each $j=1,2$) seems to be defined and related to the field in a "recurrent" way. Precisely, we have the following features: the coefficient of one function of the sequence in the expression of the term of order ε^p of the field is the same as the coefficient of the preceding function in the expression of the term of order ε^{p-1}, for each $p \geq 2$. Furthermore, we found that the T dependence of f_1 and f_2 is the same as the one of g_1 and g_2, respectively. Thus we expect the τ dependence of f_1 and f_2 to be the same as the one of g_1 and g_2, respectively, and we assume that it is truly the case. Thus we assume u_1 to be a function of Y and u_2 of X only, and the corresponding quadratures become trivial.

$\int_{X_0(Y)}^{X}[a_2(X')]^2dX'$ and $\int_{Y_0(X)}^{Y}[a_1(Y')]^2dY'$ are, however, nontrivial quadratures. We want to express these terms as integrals from 0 to x of adequate functions; the change of variable that leads to such an expression is easily determined. We find that

$$\int_{X_0(Y)}^{X}[a_2(X')]^2dX' = \frac{-2\hat{\omega}}{V_1}\int_0^x a_2^2\Big|_{-2\hat{\omega}/V_1(x'-x)+x-V_2t}dx' \quad \text{(A34)}$$

for the choice of

$$X_0(Y)=\frac{V_2}{V_1}(x-V_1t)\,, \quad \text{(A35)}$$

with analogous expressions for the second integral.

Thus the phase $\alpha_1+\beta_1T$ of g_1 can be written (in order to faciliate the determination of the symmetric expressions for the second wave, we use in following formulas the parameter η without replacing it by -1)

$$\alpha_1+\beta_1T=\frac{-x}{V_1}\left\{\frac{\eta m}{\omega_0^2}\mathcal{C}(\hat{a}_1^2+2\hat{a}_1\mathrm{Re}\,\hat{u}_1-\hat{\rho}_1^2-\hat{\rho}_2^2)\right.$$
$$\left.+\frac{m^2}{\omega_0^3}\left[\mathcal{A}\hat{a}_1^2+\mathcal{B}\frac{1}{x}\int_0^x\hat{a}_2^2\Big|_{2\eta\hat{\omega}/V_1(x'-x)+x-V_2t}dx'-\mathcal{D}\hat{\rho}_1^2-\mathcal{E}\hat{\rho}_2^2\right]\right\}+\alpha_1^{0\prime}(x-V_1t)\,, \quad \text{(A36)}$$

where $\alpha_1^{0\prime}$ is an arbitrary function of $x-V_1t$.

Furthermore, we see that

$$\|\vec{H}_1\|_{\mathrm{eff}}^2=4(\hat{a}_1^2+2\hat{a}_1\mathrm{Re}\;\hat{u}_1)+\frac{2\eta m}{\omega_0}(1+\alpha)\frac{\sin^2\theta}{\cos\theta}\hat{a}_1^2+O(\varepsilon^4)\,. \quad \text{(A37)}$$

Neglecting the terms of order ε^6 in Eq. (A36), we can replace $\hat{a}_1^2+2\hat{a}_1\mathrm{Re}\;\hat{u}_1$ using (A37) and $\hat{a}_1^2$ by $\frac{1}{4}\|\vec{H}_1\|_{\mathrm{eff}}^2$ in the term proportional to $1/\omega_0^3$. We obtain

$$\alpha_1+\beta_1T=-x\left\{\frac{\eta m}{4\omega_0^2}\mathcal{C}(\|\vec{H}_1\|_{\mathrm{eff}}^2-\|\vec{H}_1\|_\infty^2-\|\vec{H}_2\|_\infty^2)\right.$$
$$+\frac{m^2}{4\omega_0^3}\left[\mathcal{A}'\|\vec{H}_1\|_{\mathrm{eff}}^2-\mathcal{D}'\|\vec{H}_1\|_\infty^2-\mathcal{E}'\|\vec{H}_2\|_\infty^2\right.$$
$$\left.\left.+\mathcal{B}\frac{1}{x}\int_0^x\|\vec{H}_2\|_{\mathrm{eff}}^2\Big|_{2\eta\hat{\omega}/V_1(x'-x)+x-V_2t}dx'\right]+O(\varepsilon^6)\right\}+\alpha_1^{0\prime}(x-V_1t)\,, \quad \text{(A38)}$$

where, for $j=1,2$,

$$\|\vec{H}_j\|_\infty^2=\lim_{\xi\to-\infty}\|\vec{H}_j\|_{\mathrm{eff}}^2$$
$$=4\hat{\rho}_j^2+O(\varepsilon^3) \quad \text{(A39)}$$

and the constants $\mathcal{A}'$, $\mathcal{D}'$, and $\mathcal{E}'$ are deduced from $\mathcal{A}$, $\mathcal{D}$, and $\mathcal{E}$ using the coefficients that appear in expression (A37).

Analogous expressions are obtained for the second wave. The final result is summarized in Eqs. (52)–(56).

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